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Descent equations of Yang–Mills anomalies in noncommutative geometry

Edwin Langmann

Theoretical Physics, Royal Institute of Technology, S-10044 Stockholm, Sweden

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Abstract

Consistent Yang–Mills anomalies $\int \omega^{k-1}_{2n-k}$ ($n \in \mathbb{N}$, $k = 1, 2, \ldots, 2n$) as described collectively by Zumino's descent equaitons $\delta\omega^{k-1}_{2n-k} + \mathrm{d}\omega^{k}_{2n-k-1} = 0$ starting with the Chern character $Ch_{2n} = \mathrm{d}\omega^{0}_{2n-1}$ of a principal SU(N) bundle over a $2n$-dimensional manifold are considered (i.e. $\int \omega^{k-1}_{2n-k}$ are the Chern–Simons terms ($k = 1$), axial anomalies ($k = 2$), Schwinger terms ($k = 3$), etc. in $(2n - k)$ dimensions). A generalization in the spirit of Connes' non-commutative geometry using a minimum of data is found. For an arbitrary graded differential algebra $\Omega = \otimes_{k=0}^{\infty}\Omega^{(k)}$ with exterior differentiation d, form-valued functions

$$Ch_{2n} : \Omega^{(1)} \to \Omega^{(2n)}$$

and

$$\omega^{k-1}_{2n-k} : \underbrace{\Omega^{(0)} \times \cdots \times \Omega^{(0)}}_{(k-1)\text{ times}} \times \Omega^{(1)} \to \Omega^{(2n-k)}$$

are constructed which are connected by generalized descent equations $\delta\omega^{k-1}_{2n-k} + \mathrm{d}\omega^{k}_{2n-k-1} = (\cdots)$. Here $Ch_{2n} = (F_A)^n$, where $F_A = \mathrm{d}(A) + A^2$ for $A \in \Omega^{(1)}$, and $(\cdots)$ is not zero but a sum of graded commutators which vanish under integrations (traces). The problem of constructing Yang–Mills anomalies on a given graded differential algebra is thereby reduced to finding an interesting integration $\int$ on it. Examples for graded differential algebras with such integrations are given, and thereby noncommutative generalizations of Yang–Mills anomalies are found.

Subj. Class.: Noncommutative geomerty; Classical field theory
Keywords: Noncommutative geometry; Anomalies

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1. Introduction

Connes' noncommutative geometry (NCG) is a very influential subject in mathematics unifying notions and ideas from geometry and functional analysis [C1,C2] (for review see [C3]). Since recently it is also receiving broad attention from the physics community. This was triggered by a new geometric interpretation of the structure of the standard model including Higgs sector on the classical level based on NCG [CL] (see also [C3] and references therein). Even more interesting, however, is the possibility that NCG could provide a natural mathematical framework for *quantum* field theory, considering that a deeper mathematical understanding of regularization and renormalization of $(3+1)$-dimensional quantum gauge theories like QCD has been a major challenge in theoretical physics up to this day. Progress in this direction should not only lead to a better understanding of the nonperterburbative structure of these fundamental theories of nature but also provide new calculation tools for analyzing them.

One hint that NCG could be useful in this context is that one of the most widely used regularization schemes for quantum gauge theories – dimensional regualization [tHV] – is done by formally extending the models to $(4-\epsilon)$-dimentional space–time, and that NCG provides the means to naturally extend the setting of Yang–Mills gauge theories – connections on a principal $\mathrm{SU}(N)$-bundle over a manifold M – to more general situations without underlying manifold. In fact, all that is required is a graded differential algebra (GDA) (Ω, d) which is a differential complex, $\Omega = \oplus_{k=0}^{\infty}\Omega^{(k)}$ with a differentiation $d : \Omega \to \Omega$ mapping $\Omega^{(k)}$ to $\Omega^{(k+1)}$ such that $d^2 = 0$, carring a compatible algebra structure (for precise definitions see Section 2). Then elements $A \in \Omega^{(1)}$ can be interpreted as generalized Yang–Mills connections, and $X \in \Omega^{(0)}$ as infinitesimal gauge transformations acting on $\Omega^{(1)}$ by $A \to d(X) + [A, X]$. To formulate also the Yang–Mills action one needs in addition an integration $\int$ and a Hodge-$*$ operation on Ω. In case of ordinary Yang–Mills theory, this GDA is, of course, given by the de Rham forms on the space–time (or space) manifold M, but there are interesting examples for GDA (Ω, d) based on algebras of Hilbert space operators and which naturally generalize the de Rham forms [C1,C2] (see also Section 2).

One important aspect of quantized gauge theories is the occurrence of anomalies and other topological terms like axial anomalies, Schwinger terms, or Chern characters (for review see e.g. [J]). An essential requirement to any regularization procedure used in this context is that it correctly generates the anomalies. This is nontrivial – we recall the well-known difficulties to derive, or even formulate, anomalies (topological terms) in lattice gauge theories (see e.g. [$L\ddot{u}$]). The motivation for the present work was an example where a genealized Yang–Mills setting (Ω, d), as discussed above, has been used successfully for regularizing certain fermion Yang–Mills systems and giving a mathematically rigorous quantum field theory derivation of Schwinger terms [MR,M,L1,LM1] (for a more leisurely discussion of the following see [L3]). One basic idea here is to consider not only the gauge group $\mathcal{G}$ and the set of Yang–Mills fields $\mathcal{A}$, but certain algebras of Hilbert space operators in which $\mathcal{G}$ and $\mathcal{A}$ can be natural embedded. The defining relations of these algebras are Schatten ideal conditons which can be regarded as abstract characterizations of the degree of ultraviolet divergences occurring in the space–time dimension under consideration. Interestingly, this leads exactly to a GDA $(\Omega,\, \mathrm{d})$ which plays a fundamental role in NCG [C1]. One is thereby naturally lead to a universal Yang–Mills setting based on such GDA of Hilbert space operators (for another application of this in quantum field theory see [R,FR]). One advantage of using this extended setting for regularization is that it forces one to concentrate on essentials. This leads to very transparent, despite more general, constructions. One remarkable result is a universal Schwinger term in (3 + 1) dimensions [MR] generalizing the Schwinger term occurring in the Gauss' law commutators in chiral QCD(3 + 1) [LM1]. This former Schwinger terms has a natural interpretation as generalization of the latter to NCG if one regards the (conditional) Hilbert space trace as generalization of integration of de Rham forms [LM1,L2]. Having this one example where NCG appears in a quantum field theory derivation of anomalies, it is natural to ask whether other anomalies also have similar generalizations to the universal Yang–Mills setting, or, more generally, to other GDA. The structure of these generalized anomalies can be expected to directly hint at how they arise from regularizing a quantum gauge theory. For example, the regularization procedure used in [L1] to derive the anomalous Gauss' law commutators in chiral QCD trivially generalizes from (3 + 1) to higher dimensions once the higher-dimensional universal Schwinger terms are known.

In this paper we give a positive answer to this question treating all kinds of anomalies and arbitrary GDA at once. We show that certain 'raw data' for all anomalies and topological terms can be constructed in *any* GDA $(\Omega,\, \mathrm{d})$, and these give rise to anomalies (nontrivial cocycles) if and only if there is an appropriate nontrivial integration on $(\Omega,\, \mathrm{d})$. Since the GDA associated with the universal Yang–Mills setting have such nontrivial integrations (the conditional Hilbert space trace), we obtain the universal generalizations of all anomalies as a special case (including, of course, all higher-dimensional universal Schwinger terms).

We believe that many (perhaps all) interesting gauge invariant regularization schemes for Yang–Mills systems can be associated with some GDA as described in this paper. Thus our result should be fundamental for quantum gauge theories. It also provides an explanation how anomalies with their rich differential geometric structure can arise from explicit field theory calculations. The latter are done by calculating Feynman diagrams (see [J] and references therein) which can be interpreted as (regularized) traces of certain Hilbert space operators [LM2]; our result shows how the differential geometric structure of anomalies can be present already on the level of Hilbert space operators. We feel that this adds strong support to the expectation that NCG is relevant for quantum gauge theories.

To be more specific, we recall Zumino's descent equations $\delta\bar{\omega}^{\,k-1}_{2n-k} + \mathrm{d}\bar{\omega}^{\,k}_{2n-k-1} = 0$ [Z] providing a collection of intriguing relations between all Yang–Mills anomalies (the reason for the bar over ω will become clear immediately). These relations imply that the integrals $\int_M \bar{\omega}^{\,k-1}_{2n-k}$ over $(2n-k)$-dimensional manifolds M without boundary are $(k-1)$-cocycles, and these are exactly the Chern–Simons terms ($k = 1$), axial anomalies ($k = 2$), Schwinger terms ($k = 3$), etc., occurring in $(2n-k)$-dimensional gauge theories. Our result is a generalization of Zumino's descent equations to all GDA $(\Omega,\, \mathrm{d})$. For all $X_i \in \Omega^{(0)}$ and $A \in \Omega^{(1)}$ we construct forms $\omega^{k-1}_{2n-k}(X_1, \dots, X_{k-1}; A) \in \Omega^{(2n-k)}$ which are multilinear and antisymmetric in the X_i and obey the generalized descent equaitons

$$\delta\omega^{k-1}_{2n-k} + \mathrm{d}\omega^{k}_{2n-k-1} = (\cdots) \quad k = 1, 2, \ldots, 2n, \quad (n \in \mathbb{N}) \tag{1}$$

(for definition of δ see (23), for full equations (29a)–(29e)) where $(\cdots)$ is not zero but a sum of graded commutators which vanish under integrations $\int$. It is interesting to note that these descent equations generalize Zumino's even for the case of de Rham forms: we recall that the forms mentioned above are traces of matrix-valued forms, $\bar{\omega}^{k-1}_{2n-k} = \mathrm{tr}_N(\omega^{k-1}_{2n-k})$, and our equations are without the matrix trace tr_N (the terms $(\cdots)$ vanish under tr_N). Our generalization is natural since tr_N, which we regard as partial integration, is not available for general GDA.

As mentioned, these forms ω^{k-1}_{2n-k} are raw data from which the anomalies are obtained by integration $\int$; the descent equations and basic properties of integration guarantee that they obey the appropriate consistency conditions $\delta \int \omega^{k-1}_{2n-k} = 0$ (cocycle relations). Thus our result reduces the question of existence of anomalies for general GDA (Ω, d) to existence of a nontrivial integration $\int$ on it. We stress that this latter question is still very nontrivial. In fact, our result makes explicit that all topological information is contained in such an integration: the raw data for topological terms ω^{k-1}_{2n-k} *always* exist and give rise to $(k-1)$-cocyles by integration $\int$; whether these are nontrivial solely depends on the integration $\int$ and has to be checked for each case individually.

We note that there are other generalizations of Zumino's descent equations in the literature [CM,KS]. These generalize the equation for $\bar{\omega}^{k-1}_{2n-k}$ and therefore also need the notation of (partial) integration. Our result is more general since it does not require an integration. In fact, we believe that it is the most general result in this direction and very much in the spirit of NCG since it uses a minimum amount of input data and strips the descent equations from a differential geometric setting to the bones of simple algebraic relations. (Note also that we do not have a graded double complex since $\delta\mathrm{d} + \mathrm{d}\delta$ is not zero on forms ω^{k-1}_{2n-k} but only on $\bar{\omega}^{k-1}_{2n-k}$!) Also, our proof is very simple and direct – we give an explicit formula for the forms ω^{k-1}_{2n-k}, and prove by explicit calculation that our generalized descent equations are obeyed. We did not attempt to find a geometric interpretation of these equations (1), but since the nontrivial RHS $(\cdots)$ have a very simple, suggestive form (cf. (29a)–(29e)), we believe that such an interpretation is possible and should be interesting.

There are two important examples for GDA of bounded Hilbert space operators which generalize the de Rham forms to NCG. In one, differentiation is given by the graded commutator with the Dirac operator and integration by Dixmier trace [C2], and in the other, differentiation is graded commutation with the *sign* of the Dirac operator and integration is the Hilbert space trace [C1] (see also [C3]). In this paper we use only the latter as example since it is this GDA which naturally appears in quantum field theory. Also, the noncommutative generalization of anomalies to the former is discussed in [CM].

To avoid confusion, we already point out here that we use Grassmann numbers θ_i for writing down the formulas for the forms ω^{k-1}_{2n-k}, but this is only a matter of convenience and one could do without at the cost of having longer expressions and proofs. However, these Grassmann numbers are a convenient means to keep track of the combinatorics and order of terms – they are a substitute for the symmetrized trace used by Zumino [Z] which is not available in our general setting.

The plan of this paper is as follows. To make the paper self-contained and have nontrivial examples to which our result can be applied, we summarize the basis definitions and examples of GDA and integrations in Section 2. This section is also meant as an introduction to basic notions from NCG [C3] relevent for quantum field theory. Of mathematical interest might be our definition of partial integration on GDA (generalizing integration of de Rham forms over manifolds with boundary) which seems somewhat different form the one in [C3]. Our main result and its proof are in Section 3. In Section 4 we discuss some interesting special cases including the universal anomalies mentioned.

Notation. We denote as $\mathbb{N}_0$ the nonnegative integers, $\mathbb{C}$ the complex numbers, gl_N the complex algebra of $N \times N$ matrices with algebra product the usual matrix multiplication, $[a, b]_\pm = ab \pm ba$, $[a, b] = ab - ba$ and $\{a, b\} = ab + ba$.

2. Prerequisites

2.1. Graded differential algebra

A graded differential algebra (GDA) (Ω, d) is a $\mathbb{N}_0$-graded associative algebra over $\mathbb{C}$, $\Omega = \oplus_{k=0}^{\infty}\Omega^{(k)}$, with a linear map $\mathrm{d} : \Omega \to \Omega$ such that $\mathrm{d}^2 = 0$, and for all $\omega_{k,l} \in \Omega^{c(k,l)}$, $k, l \in \mathbb{N}_0$,

$$\omega_k\omega_l \in \Omega^{(k+l)}, \quad \mathrm{d}\omega_k \in \Omega^{(k+1)}, \quad \mathrm{d}(\omega_k\omega_l) = \mathrm{d}(\omega_k)\omega_l + (-)^k\omega_k\,\mathrm{d}(\omega_l) \tag{2}$$

(we write the algebra product in Ω as $(\omega, \omega') \to \omega\omega'$).

We denote elements $\omega_k \in \Omega^{(k)}$ as k-forms and d as exterior differentiation. Note that by definition, all linear combinations of elements $u_0\,\mathrm{d}u_1 \cdots \mathrm{d}u_k$ with $u_i \in \Omega^{(0)}$ are k-forms.

Example 1. The simplest example for a GDA is the de Rham GDA $(\Omega_{\mathrm{dR},d}, \mathrm{d})$ of de Rham forms on $\mathbb{R}^d$ which we define as follows: $\Omega^{(0)}_{\mathrm{dR},d} = C_0^\infty(\mathbb{R}^d; \mathrm{gl}_N)$ is the algebra of gl_N-valued C_0^∞ functions on $\mathbb{R}^d$, and for $k \geq 1$, $\Omega^{(k)}_{\mathrm{dR},d}$ is the vector space generated by elements $u_0\,\mathrm{d}u_1 \cdots \mathrm{d}u_n$, $u_i \in \Omega^{(0)}_{\mathrm{dR},d}$, where d is the usual exterior differentiation of de Rham forms.

Example 2. Let $\mathcal{H}$ be a separable Hilbert space. We denote as $\mathcal{B}$ and $\mathcal{B}_1$ the bounded and trace class operators on $\mathcal{H}$, respectively, and $\mathcal{B}_p = \{a \in \mathcal{B} \,|\, (a^*a)^{p/2} \in \mathcal{B}_1\}$ for $p > 0$ are the Schatten classes. We recall that $a \in \mathcal{B}_{n/p}$, $b \in \mathcal{B}_{n/q}$ and $n, p, q > 0$ implies $ab \in \mathcal{B}_{n/(p+q)}$ [S]. Let $\varepsilon \in \mathcal{B}$ be a grading operator, i.e. $\varepsilon = \varepsilon^* = \varepsilon^{-1}$. We will use the notation

$$a = a_+ + a_-, \quad a_\pm = \tfrac{1}{2}(a \pm \varepsilon a \varepsilon) \quad \forall a \in \mathcal{B} \tag{3}$$

so that $(ab)_+ = a_+b_+ + a_-b_-$, $(ab)_- = a_+b_- + a_-b_+$.

Let $d \in \mathbb{N}$. We define

$$p = \tfrac{1}{2}(d+1) \tag{4a}$$

and

$$\begin{aligned} \hat{\Omega}_d^{(0)} &= \{a \in \mathcal{B} \mid a_- \in \mathcal{B}_{2p}\}, \\ \hat{\Omega}_d^{(2k-1)} &= \{a \in \mathcal{B} \mid a_+ \in \mathcal{B}_{2p/2k},\ a_- \in \mathcal{B}_{2p/(2k-1)}\}, \\ \hat{\Omega}_d^{(2k)} &= \{a \in \mathcal{B} \mid a_+ \in \mathcal{B}_{2p/2k},\ a_- \in \mathcal{B}_{2p/(2k+1)}\}, \\ k &= 1, 2, \ldots \end{aligned} \tag{4b}$$

Then

$$\hat{\mathrm{d}}\omega_k \equiv \mathrm{i}(\varepsilon\omega_k - (-)^k \omega_k \varepsilon) \quad \forall \omega_k \in \hat{\Omega}_d^{(k)} \tag{4c}$$

defines an exterior differentiation $\hat{\mathrm{d}}$ on $\hat{\Omega}_d = \oplus_{k=0}^{\infty} \hat{\Omega}_d^{(k)}$ which makes $(\hat{\Omega}_d, \hat{\mathrm{d}})$ into a GDA.

Note that the algebra multiplication in this GDA is equal to the operator product in $\mathcal{B}$ (the latter follows from $\varepsilon^2 = 1$ which implies $\varepsilon[\varepsilon, a]\varepsilon = -[\varepsilon, a]$ for all $a \in \mathcal{B}$).

For $d = 2n$ *even*, we introduce another grading operator Γ on $\mathcal{H}$ which anticommutes with ε, $\varepsilon\Gamma = -\Gamma\varepsilon$. For the introduction of integrations below we need to restrict $(\hat{\Omega}_d, \hat{\mathrm{d}})$ to a subalgebra $(\hat{\Omega}_{\mathrm{dR},d}, \hat{\mathrm{d}})$ defined as $\hat{\Omega}_{\mathrm{dR},d} = \oplus_{k=0}^{\infty} \hat{\Omega}_{\mathrm{dR},d}^{(k)}$,

$$\hat{\Omega}_{\mathrm{dR},d}^{(k)} = \{\omega_k \in \hat{\Omega}_d^{(k)} \mid \omega_k \Gamma - (-)^k \Gamma \omega_k = 0\} \quad (d = 2n) \tag{5}$$

(it is easy to see that this indeed defines a GDA). To simplify notation we write $\Gamma^{d-1} = \Gamma$ and 1 (identity) for d even and odd, respectively, and for d odd, $\hat{\Omega}_{\mathrm{dR},d} = \hat{\Omega}_d$.

The above-mentioned embedding of the de Rham GDA $(\hat{\Omega}_{\mathrm{dR},d}, \mathrm{d})$ in $(\hat{\Omega}_{\mathrm{dR},d}, \hat{\mathrm{d}})$ is as follows. Consider the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^{v}_{\mathrm{spin}} \otimes \mathbb{C}^N, \tag{6}$$

where $v = 2^{[d/2]}$ can be interpreted as the number of spin indices ($[d/2] = \frac{1}{2}(d-1)$ for d odd and $[d/2] = \frac{1}{2}d$ for d even). With γ_i the usual self-adjoint γ-matrices acting on $\mathbb{C}^{v}_{\mathrm{spin}}$ and obeying $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$ (for explicit formulas see e.g. [L2]), the free Dirac operator $\not{D}_0 = \sum_{i=1}^{d} \gamma_i(-\mathrm{i})\partial/\partial x_i$ on $\mathbb{R}^d$ defines a self-adjoint operator on $\mathcal{H}$ which we also denote as $\not{D}_0$. For $d = 2n$ even we have an additional γ-matrix γ_{d+1} which defines a grading operator Γ on $\mathcal{H}$.

We recall that every $X \in C_0^{\infty}(\mathbb{R}^d, \mathrm{gl}_N)$ can be naturally identified with a bounded operator $\hat{X}$ on $\mathcal{H}$, $(\hat{X}f)(x) = X(x)f(x)$ for all $f \in \mathcal{H}(x \in \mathbb{R}^d)$.

We define $\varepsilon = \mathrm{sign}(\not{D}_0)$ where $\mathrm{sign}(x) = +1(-1)$ for $x \geq 0$ ($x < 0$) (using the spectral theorem for self-adjoint operators [RS]). For $d = 2n$ even, $\Gamma = \gamma_{d+1}$ anticommutes with ε, and it obviously commutes with all $\hat{X}$ for $X \in C_0^{\infty}(\mathbb{R}^d, \mathrm{gl}_N)$.

One can prove

$$X \in C_0^{\infty}(\mathbb{R}^d, \mathrm{gl}_N) \quad \Rightarrow \quad \hat{X} \in \hat{\Omega}_{\mathrm{dR},d}^{(0)} \tag{7a}$$

(see e.g. [MR]) implying that

$$X_0\,\mathrm{d}X_1\cdots X_k \mapsto (i)^k\hat{X}_0[\varepsilon,\hat{X}_1]\cdots[\varepsilon,\hat{X}_k]\quad \forall X_i\in\Omega^{(k)}_{\mathrm{dR},d} \tag{7b}$$

provides an embedding $\omega\mapsto\hat{\omega}$ of $(\Omega_{\mathrm{dR},d},\ \mathrm{d})$ in $(\hat{\Omega}_{\mathrm{dR},d},\ \hat{\mathrm{d}})$

We shall refer to $(\hat{\Omega}_{\mathrm{dR},d},\ \hat{\mathrm{d}})$ as *universal de Rham GDA*.

2.2. *Integrations*

A GDA with integration is a triple $(\Omega,\ \mathrm{d},\ \int)$ such that $(\Omega,\ \mathrm{d})$ is a GDA, and there is a linear map $\int:\Omega\to\mathbb{C}$ obeying

$$\int \mathrm{d}\omega_k = 0,\quad \int\omega_k\omega_l = (-)^{kl}\int\omega_l\omega_k\qquad \forall\omega_{k,l}\in\Omega^{(k,l)}. \tag{8}$$

(The first of these relations is Stokes' theorem and the second graded commutativity of forms under the integration.) If $\int$ in nonzero only on $\Omega^{(d)}$ for some $d\in\mathbb{N}_0$ we say that Ω has dimension d and indicate this by writing $\Omega=\Omega_d$.

Remark. A GDA with integration in called *cycle* in [C3].

Example 3. The de Rham GDA $(\Omega_{\mathrm{dR},d},\ \mathrm{d})$ has the natural integration defined as $\int\omega_k=0$ for $\omega_k\in\Omega^{(k\neq d)}_{\mathrm{dR},d}$ and

$$\int\omega_d = \int_{\mathbb{R}^d}\mathrm{tr}_N(\omega_d)\quad \forall\omega_d\in\Omega^{(d)}_{\mathrm{dR},d} \tag{9}$$

(integration of de Rham forms as usual; $\mathrm{tr}_N(\cdot)$ is the trace of $N\times N$ matrices).

Example 4. We define the conditional trace as

$$\mathrm{Tr}_C(a)\equiv\mathrm{Tr}(a_+)\quad\text{for all } a\in\mathcal{B}\ \text{ with } a_+\in\mathcal{B}_1, \tag{10}$$

where $\mathrm{Tr}_C(\cdot)$ is the usual Hilbert space trace on $\mathcal{H}$. Then $\hat{\int}\omega_k=0$ for all $\omega_k\in\hat{\Omega}^{(k\neq d)}_{dR,d}$ and

$$\hat{\int}\omega_d = \frac{1}{c_d}\mathrm{Tr}_C(\Gamma^{d-1}\omega_d)\quad\forall\omega_d\in\hat{\Omega}^{(d)}_{\mathrm{dR},d}, \tag{11a}$$

where

$$c_d = (2\mathrm{i})^{[d/2]}\frac{1}{d(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{11b}$$

is a convenient normalization constant (explained below), defines an integration on $(\hat{\Omega}_{\mathrm{dR},d},\ \mathrm{d})$. Indeed, for $\omega_d\in\hat{\Omega}^{(d)}_{\mathrm{dR},d}$,

$$(\Gamma^{d-1}\omega_d)_+ = \begin{cases}(\omega_d)_+ & \text{for } d \text{ odd},\\ \Gamma(\omega_d)_- & \text{for } d \text{ even},\end{cases}$$

which is $\mathcal{B}_1$ by definition. Relations (8) can be easily checked. (The first relation of (8) (Stokes' theorem) is trivial since

$$(\Gamma^{d-1}\hat{\mathrm{d}}\omega_{d-1})_+ = -\tfrac{1}{2}\mathrm{i}\Gamma^{d-1}\varepsilon\,\hat{\mathrm{d}}^2(\omega_{d-1}) = 0.$$

The second follows from

$$\begin{aligned}&(\Gamma^{d-1}(\omega_k\omega_{d-k} - (-)^{k(d-k)}\omega_{d-k}\omega_k))_+\\ &\quad= [\Gamma^{d-1}\omega_k)_+, (\omega_{d-k})_+] + [(\Gamma^{d-1}\omega_k)_-, (\omega_{d-k})_-]\end{aligned}$$

and cyclicity of trace (recall that $\mathrm{Tr}([a,b]) = 0$ if at least one of $a, b \in \mathcal{B}$ is compact and ab and ba are $\mathcal{B}_1$). Note that in case d is even, we used that $\Gamma\omega_k = (-)^k\omega_k\Gamma$; this is why we had to restrict ourselves from $\hat{\Omega}_d$ to $\hat{\Omega}_{\mathrm{dR},d}$.)

The embedding $\omega \mapsto \hat{\omega}$ of de Rham GDA $(\Omega_{\mathrm{dR},d}, \mathrm{d})$ in $(\hat{\Omega}_{\mathrm{dR},d}, \hat{\mathrm{d}})$ as discussed above naturally extends to these integrations. One has

$$\begin{aligned}&\mathrm{i}^d\mathrm{Tr}_C(\Gamma^{d-1}\hat{X}_0[\varepsilon,\hat{X}_1]\cdots[\varepsilon,\hat{X}_d])\\ &\quad= c_d\int \mathrm{tr}_N(X_0\,\mathrm{d}X_1\cdots\mathrm{d}X_d)\quad \forall X_i \in C_0^\infty(\mathbb{R}^d;\mathrm{gl}_N)\end{aligned}\tag{12}$$

(for an elementary proof see [22]), which shows that $\hat{\int}\hat{\omega} = \int\omega$ for all $\omega \in \Omega_{\mathrm{dR},d}$.

2.3. Partial integrations

A GDA with partial integration structure (GDAPI) is a quintuple $(\Omega, \mathrm{d}, \int, {}^*\Omega, \partial)$ such that: (i) (Ω, d) is a GDA; (ii) ${}^*\Omega = \oplus_{k=0}^\infty {}^*\Omega^{(-k)}$ is graded cycle, i.e. an $\mathbb{N}_0$-graded vector space with a linear map $\partial : {}^*\Omega \to {}^*\Omega$ such that $\partial Q^{(-k)} \in {}^*\Omega^{(-k-1)}$ for all $Q^{(-k)} \in {}^*\Omega^{(-k)}$ and $\partial^2 = 0$; (iii) there is a bilinear map $\int : {}^*\Omega \times \Omega \to \mathbb{C}$, $(Q,\omega) \mapsto \int_Q \omega$ such that

$$\int_Q \mathrm{d}\omega_k = \int_{\partial Q}\omega_k, \qquad \int_Q \omega_k\omega_l = (-)^{kl}\int_Q \omega_l\omega_k \tag{13}$$

for all $\omega_{k,l} \in \Omega^{(k,l)}$ and $Q \in {}^*\Omega$. (The first of these is the generalization of Stokes' theorem for manifolds with boundaries.) A GDAPI is of dimension d if $\int$ is nonzero only on $\oplus_{k=0}^d {}^*\Omega^{(-k)} \times \Omega^{(d-k)}$, and we write in this case $\Omega = \Omega_d$ and ${}^*\Omega = {}^*\Omega_d$.

In our examples below, GDAPI $(\mathcal{S}, \mathrm{d}, \int, {}^*\mathcal{S}, \partial)$ arise from a given GDA $(\Omega, \mathrm{d}, \int)$ with integration as follows: (i) $(\mathcal{S}, \mathrm{d}, \int)$ is a subcomplex of, i.e. a GDA contained in, $(\Omega, \mathrm{d}, \int)$; (ii) ${}^*\mathcal{S} = \oplus_{k=0}^\infty {}^*\mathcal{S}^{(-k)}$ is a graded complex such that for each $Q \in {}^*\mathcal{S}$ and $0 < \epsilon < 1$ there is a $\eta_\epsilon(Q) \in \Omega$ such that

$$\lim_{\epsilon\searrow 0}\int \eta_\epsilon(Q)\omega \tag{14a}$$

exist for all $\omega \in \mathcal{S}$; (iii) for all $\omega_{k,l} \in \mathcal{S}^{(k,l)}$,

$$\lim_{\epsilon\searrow 0}\int \eta_\epsilon(Q)(\omega_k\omega_l - (-)^{kl}\omega_l\omega_k) = 0. \tag{14b}$$

We then set

$$\int_Q \omega \equiv \lim_{\epsilon \searrow 0} \int \eta_\epsilon(Q)\omega, \tag{15}$$

and ∂Q is defined by

$$\eta_\epsilon(\partial Q)_{-k} \equiv (-)^{-k-1}\, \mathrm{d}\eta_\epsilon(Q)_{-k} \tag{16}$$

(we use the notation $\Omega \ni \eta = \sum_k \eta_{-k}$ with $\eta_{-k} \in \Omega^{(k)}$). Stokes' theorem in (13) then trivially follows from $\int \mathrm{d}(\eta_\epsilon(Q)\omega) = 0$. Thus it is (14b) which is the nontrivial condition determining whether $\mathcal{S}$ and $^*\mathcal{S}$ are compatible so as to form a GDAPI.

Obviously $(\mathcal{S}, \mathrm{d}, \int, {}^*\mathcal{S}, \partial)$ is of dimension d if $(\Omega, \mathrm{d}, \int)$ is.

Remark. One can identify elements in $^*\mathcal{S}$ with equivalence classes $Q = \eta/\sim$ of maps $\eta : (0,1) \to \Omega, \epsilon \mapsto \eta_\epsilon$ such that $\lim_{\epsilon\searrow 0} \int \eta_\epsilon \omega$ exist for all $\omega \in \mathcal{S}$, with the equivalence relation

$$\eta \sim \eta' \iff \lim_{\epsilon\searrow 0} \int (\eta_\epsilon - \eta'_\epsilon)\omega = 0 \;\; \forall \omega \in \mathcal{S}.$$

Example 5. Let $\mathcal{D}_d^{(0)}$ be the set of d-dimensional compact submanifolds D of $\mathbb{R}^d$ with boundary ∂D which is a $(d-1)$-dimensional compact C^∞ manifold, and $\mathcal{D}_d^{(-k)}$ be the set of all $(d-k)$-dimensional compact C^∞ manifolds of the form $D^{(-k)} = D_0 \cap \partial D_1 \cap \cdots \cap \partial D_k$ with $D_i \in \mathcal{D}_d^{(0)}$. Then $(\Omega_{\mathrm{dR},d}, \mathrm{d}, \int, \mathcal{D}, \partial)$ is a GDAPI with ∂ the usual boundary operation and $\int_{D^{(-k)}} \omega_{d-k} \equiv \int_{D^{(-k)}} \mathrm{tr}_N(\omega_{d-k})$ the integration of a $(d-k)$-form over an k-dimensional submanifold $D^{(-k)}$ of $\mathbb{R}^d$ as usual. With $\mathcal{X}_D(x)$ the characteristic function of $D \in \mathcal{D}_d^{(0)}$ ($\mathcal{X}_D(x) = 1$ for $x \in D$ and $= 0$ otherwise) we can formally write

$$\begin{aligned}\int_{D_0\cap\partial D_1\cap\cdots\cap\partial D_k} \mathrm{tr}_N(\omega_{d-k}) &= \int_{\partial D_1\cap\cdots\cap\partial D_k} \mathrm{tr}_N(\mathcal{X}_{D_0}\omega_{d-k}) \\ &= \int_{D_1\cap\partial D_2\cap\cdots\cap\partial D_k} \mathrm{tr}_N(\mathrm{d}(\mathcal{X}_{D_0}\omega_{d-k})) = \cdots \\ &= \int_{\mathbb{R}^d} \mathrm{tr}_N(\mathcal{X}_{D_k}\, \mathrm{d}\mathcal{X}_{D_{k-1}} \cdots \mathrm{d}\mathcal{X}_{D_1}\, \mathrm{d}(\mathcal{X}_{D_0}\omega_{d-k})) \\ &= (-)^k \int_{\mathbb{R}^d} \mathrm{tr}_N(\mathrm{d}\mathcal{X}_{D_k} \cdots \mathrm{d}\mathcal{X}_{D_1}\mathcal{X}_{D_0}\omega_{d-k}),\end{aligned}$$

where we used repeatedly $\partial^2 = \mathrm{d}^2 = 0$ and Stokes' theorem for integration of de Rham forms over manifolds with boundaries. Introducing approximate δ-functions $\delta^{(\epsilon)} \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ for $0 < \epsilon < 1$ obeying $\delta^{(\epsilon)}(x) \geq 0$, for all $x \in \mathbb{R}^d$, $= 0$ for $|x| \geq \epsilon$ and $\int_{\mathbb{R}^d} \mathrm{d}^d x \delta^{(\epsilon)}(x) =$

1, we define $\mathcal{X}_D^{(\epsilon)}(x) = \int_{\mathbb{R}^d} \mathrm{d}^d y \delta^{(\epsilon)}(x-y)\mathcal{X}_D(y)$. With that, our formal calculation above implies

$$\int\limits_{D^{(-k)}} \mathrm{tr}_N(\omega_{d-k}) = \lim_{\epsilon\searrow 0} \int\limits_{\mathbb{R}^d} \mathrm{tr}_N(\eta_\epsilon(D^{(-k)})\omega_{d-k}) \tag{17a}$$

with

$$\eta_\epsilon(D_0 \cap \partial D_1 \cap \cdots \cap \partial D_k) = (-)^k \,\mathrm{d}\mathcal{X}_{D_k}^{(\epsilon)} \cdots \mathrm{d}\mathcal{X}_{D_1}^{(\epsilon)} \mathcal{X}_{D_0}^{(\epsilon)} \in \Omega_{\mathrm{dR},d}^{(k)} \quad \forall \epsilon > 0, \tag{17b}$$

and we see that we have an example of a GDAPI as discussed above.

Example 6. For all $D \in \mathcal{D}^{(0)}$ and $0 < \epsilon < 1$, $\mathcal{X}_D^{(\epsilon)}$ define operators $Q_D^{(\epsilon)}$ on $\mathcal{H}$ (6), $(Q_D^{(\epsilon)} f)(x) = \mathcal{X}_D^{(\epsilon)}(x) f(x)$ for all $f \in \mathcal{H}$, and these opertors are in $\hat{\Omega}_{\mathrm{dR},d}^{(\epsilon)}$. Thus

$$\eta_\epsilon(Q^{(-k)}) = (-\mathrm{i})^k [\varepsilon, Q_{D_k}^{(\epsilon)}] \cdots [\varepsilon, Q_{D_1}^{(\epsilon)}] Q_{D_0}^{(\epsilon)} \tag{18}$$

are in $\hat{\Omega}_{\mathrm{dR},d}^{(k)}$ and define a GDAPI $(\hat{\mathcal{S}}_{\mathrm{dR},d}, \mathrm{d}, \int, {}^*\hat{\mathcal{S}}_{\mathrm{dR},d}, \partial)$ as discussed above. We do not attempt here to specify $\hat{\mathcal{S}}_{\mathrm{dR},d}$ and just note that it at least contains $\Omega_{\mathrm{dR},d}$ as follows from our discussion above. Especially Eq. (12) implies that this GDAPI is a natural noncommutative generalization of the GDAPI $(\Omega_{\mathrm{dR},d}\, \mathrm{d}, \int, \mathcal{D}, \partial)$ from Example 5. Other examples are obtained by generating ${}^*\hat{\mathcal{S}}_{\mathrm{dR},d}$ with other families of bounded, self-adjoint operators Q_i with $0 \le Q_i \le 1$ and determining $\hat{\mathcal{S}}_{\mathrm{dR},d}$ such that (14b) holds.

3. Noncommutative descent equations

3.1. Result

Definitions. Given a GDA (Ω, d), we can interpret $\Omega^{(1)}$ as generalized Yang–Mills connections and $\Omega^{(0)}$ as Lie algebra of the gauge group acting on $\Omega^{(1)}$ as

$$\Omega^{(1)} \times \Omega^{(0)} \to \Omega^{(1)}, \qquad (A, X) \mapsto \mathrm{d}(X) + [A, X],$$

and on all polynomial functions $f : \Omega^{(1)} \to \Omega,\ A \to f(A)$ by Lie derivative

$$\mathcal{L}_X f(A) \equiv \frac{\mathrm{d}}{\mathrm{d}t} f(A + t(\mathrm{d}(X) + [A, X]))|_{t=0} \quad \forall X \in \Omega^{(0)}. \tag{19}$$

We also define a modified Lie derivative

$$\hat{\mathcal{L}}_X f(A) \equiv \mathcal{L}_X f(A) - [f(A), X], \tag{20}$$

which obviously also obeys the Leibnitz rule, $\hat{\mathcal{L}}_X(ff') = \hat{\mathcal{L}}_X(f)f' + f\hat{\mathcal{L}}_X(f')$.

In the following we consider k-chains which we define as polynomial functions

$$f^k : \underbrace{\Omega^{(0)} \times \cdots \times \Omega^{(0)}}_{k \text{ times}} \times \Omega^{(1)} \to \Omega,$$

$$(X_1 \dots, X_k, A) \mapsto f^k(X_1, \dots, X_k; A), \tag{21}$$

which are multilinear in the first k arguments, and antisymmetric, i.e. for all permutations π of $(1, \dots, k)$,

$$f^k(X_{\pi(1)}, \dots, X_{\pi(k)}; A) = (-)^{\deg(\pi)} f^k(X_1, \dots, X_k; A), \tag{22}$$

where $\deg(\pi)$ is 0 and 1 for even and odd permutations π, respectively. On such chains we define the operator

$$\begin{aligned} &\delta f^{k-1}(X_1, \dots, X_k; A) \\ &\quad = \sum_{j=1}^{k} (-)^{j+1} \mathcal{L}_{X_j} f^{k-1}(X_1, \dots, \not{X}_j, \dots, X_k; A) \\ &\qquad + \sum_{\substack{i,j=1 \\ i<j}}^{k} (-)^{i+j} f^{k-1}([X_i, X_j], X_1, \dots, \not{X}_i, \dots, \not{X}_j, \dots, X_k; A), \end{aligned} \tag{23}$$

where $\not{X}_j$ means that X_j is omitted. δ is the usual operation acting on chains and satisfying $\delta^2 = 0$. We also introduce the operator $\hat{\delta}$ which is defined as in Eq. (23) but with $\hat{\mathcal{L}}_{X_j}$ (20) instead of $\mathcal{L}_{X_j}$. We write this operator as $\hat{\delta} = \delta - \mathcal{J}$ with $\mathcal{J}$ the operator acting on chains as

$$\mathcal{J} f^{k-1}(X_1, \dots, X_k; A) = \sum_{j=1}^{k} (-)^{j+1} [f^{k-1}(X_1, \dots, \not{X}_j, \dots, X_k; A), X_j]. \tag{24}$$

For positive, even integers $2n$, we define the $2n$-forms

$$Ch_{2n}(A) \equiv (F_A)^n, \quad F_A \equiv \mathrm{d}(A) + A^2 \quad \forall A \in \Omega^{(1)}; \tag{25}$$

F_A is the curvature associated with the Yang–Mills connection A.

To write our formulas in compact form we introduce Grassmann variables θ_i and integration $\int \mathrm{d}\theta_i$ over them satisfying usual relations

$$\begin{aligned} &\{\theta_i, \theta_j\} = \{\theta_i, \mathrm{d}\theta_j\} = \{\mathrm{d}\theta_i, \mathrm{d}\theta_j\} = [\theta_i, \omega] = [\mathrm{d}\theta_i, \omega] = 0, \\ &\int \mathrm{d}\theta_i \theta_j = \delta_{ij}, \quad \int \mathrm{d}\theta_i = 0 \qquad \forall i, j = 0, 1, 2, \dots, 2n, \quad \omega \in \Omega. \end{aligned} \tag{26}$$

For $A \in \Omega^{(1)}$, $X_i \in \Omega^{(0)}$, $0 \le t \le 1$, we set

$$\begin{aligned} &\upsilon^k_{2n-k-1}(X_1, \dots, X_k; A, t) \\ &\quad \equiv \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \left(F_{tA} + \theta_0 A + \sum_{i=1}^{k} (t-1)\theta_i \, \mathrm{d}(X_i) \right)^n, \\ &\quad k = 0, 1, \dots, n-1, \end{aligned} \tag{27a}$$

where $F_{tA} = t\mathrm{d}(A) + t^2A^2$, and

$$
\begin{aligned}
&v^k_{2n-k-1}(X_1, \dots, X_k; A, t) \\
&\quad\equiv (-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \\
&\quad\times \left(\sum_{i=1}^{k} t\theta_i \,\mathrm{d}(X_i) + \sum_{\substack{i,j=1\\ i<j}}^{k} (t^2 - t)\theta_i\theta_j [X_i, X_j] + \sum_{i=1}^{k} \theta_0\theta_i X_i \right)^n , \\
&\quad k = n, n+1, \dots, 2n-1,
\end{aligned} \tag{27b}
$$

(note that the v^k_{2n-k-1} for $k > n$ are actually independent of A). Here and in the following we write $\int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0$ short for $\int \mathrm{d}\theta_k \int \mathrm{d}\theta_{k-1} \cdots \int \mathrm{d}\theta_0$.

Using this we then define k-chains

$$
\omega^k_{2n-k-1}(X_1, \dots, X_k; A) \equiv \int_0^1 \mathrm{d}t \; v^k_{2n-k-1}(X_1, \dots, X_k; A, t), \tag{28a}
$$

$$
\tilde{\omega}^k_{2n-k-1}(X_1, \dots, X_k; A) \equiv \int_0^1 \mathrm{d}t \; t v^k_{2n-k-1}(X_1, \dots, X_k; A, t), \tag{28b}
$$

$$
k = 0, 1, \dots, 2n-1
$$

which are easily seen to be forms of the degrees indicated (this follows from (26)).

Theorem. For arbitrary GDA (Ω, d) and $n = 1, 2, \dots$, the forms defined above obey the following generalized descent equations:

$$
Ch_{2n} = \mathrm{d}(\omega^0_{2n-1}) + \{A, \tilde{\omega}^0_{2n-1}\}, \tag{29a}
$$

$$
\begin{aligned}
&\delta\omega^{k-1}_{2n-k} + \mathrm{d}(\omega^k_{2n-k-1}) = \mathcal{J}\omega^{k-1}_{2n-k} - [A, \tilde{\omega}^k_{2n-k-1}]_\sigma, \\
&\quad \sigma = (-)^k, \quad k = 1, 2, \cdots n-1,
\end{aligned} \tag{29b}
$$

$$
\delta\omega^{n-1}_n + \mathrm{d}(\omega^n_{n-1}) = \mathcal{J}\omega^{n-1}_n, \tag{29c}
$$

$$
\delta\omega^{k-1}_{2n-k} + \mathrm{d}(\omega^k_{2n-k-1}) = \mathcal{J}\tilde{\omega}^{k-1}_{2n-k}, \quad k = n+1, \dots, 2n-1, \tag{29d}
$$

$$
\delta\omega^{(2n-1)}_0 = \mathcal{J}\tilde{\omega}^{(2n-1)}_0. \tag{29e}
$$

(We recall that $[a, b]_\sigma = ab + \sigma ba$ for $\sigma = \pm$.)

Corollary. For all differential graded algebras with partial integration structure (GDAPI) $(\Omega, \mathrm{d}, \int, {}^*\Omega, \partial)$ the forms defined above obey the following relations:

$$
\begin{aligned}
&\delta \int_Q \omega^{k-1}_{2n-k} = \int_{\partial Q} \omega^k_{2n-k-1}, \\
&\forall k = 1, 2, \dots, 2n, \quad n = 1, 2, \dots, \quad Q \in {}^*\Omega
\end{aligned} \tag{30}
$$

(we set $\omega^{2n}_{-1} \equiv 0$, and $\delta \int_Q \equiv \int_Q \delta$).

3.2. Proof

We will divide the proof in three parts (a)–(c).

For the following we note that every element $\omega_k \in \Omega^{(n)}$ naturally defines an operator $L_{\omega_k}: \Omega \to \Omega, \omega \mapsto L_{\omega_k}(\omega) \equiv \omega_k\omega$ (left multiplication), and obviously $\mathrm{d}L_{\omega_k} = L_{\mathrm{d}\omega_k} + (-)^k L_{\omega_k}\mathrm{d}$. Since the distinction of L_{ω_k} from ω_k would clutter notation, we simply write $L_{\omega_k} = \omega_k$. The former relation then is written as $\mathrm{d}\omega_k = \mathrm{d}(\omega_k) + (-)^k\omega_k\,\mathrm{d}$. It is therefore important in the following to distinguish between $\mathrm{d}\omega$ and $\mathrm{d}(\omega)$ ($\omega \in \Omega$). Thus for $A \in \Omega^{(1)}$, $\mathrm{d}(A) = \{\mathrm{d}, A\}$, and since $\mathrm{d}^2 = 0$ we can write $F_A = (\mathrm{d} + A)^2$. In this notation Bianchi identity is trivial, $[\mathrm{d} + A, F_A] = 0$. Similarly for $X \in \Omega^{(0)}$, $\mathrm{d}(X) = [\mathrm{d}, X]$.

(a) We write ($\partial_t \equiv \partial/\partial t$)

$$Ch_{2n}(A) = \int_0^1 \mathrm{d}t\,\partial_t(F_{tA})^n = \int_0^1 \mathrm{d}t \sum_{v=0}^{n-1}(F_{tA})^{n-1-v}\partial_t(F_{tA})(F_{tA})^v$$

(note that $F_0 = 0$). Since $\partial_t(F_{tA}) = \mathrm{d}(A) + 2tA^2 = \{\mathrm{d} + tA, A\}$ and $\mathrm{d} + tA$ commutes with F_{tA} (Bianchi identity), this is equivalent to

$$Ch_{2n}(A) = \int_0^1 \mathrm{d}t \left\{\mathrm{d} + tA, \sum_{v=0}^{n-1}(F_{tA})^{n-1-v}A(F_{tA})^v\right\}.$$

We note that (cf. (27a))

$$v^0_{2n-1}(A;t) = \int \mathrm{d}\theta_0(F_{tA} + \theta_0 A)^n = \sum_{v=0}^{n-1}(F_{tA})^{n-1-v}A(F_{tA})^v, \tag{31}$$

and with our definition (28a) we obtain (29a).

(b) Here $k = 1, 2, \ldots, n$. We shall use

$$\mathcal{L}_X(A) = [\mathrm{d} + A, X], \qquad \mathcal{L}_X(F_A) = [F_A, X]. \tag{32}$$

This implies

$$\mathcal{L}_X(F_{tA}) = [F_{tA}, X] + (t-1)\{\mathrm{d} + tA, [\mathrm{d}, X]\}. \tag{33}$$

(To see this, note that $F_{tA} = tF_A + (t^2 - t)A^2$, thus

$$\begin{aligned}\mathcal{L}_X(F_{tA}) &= t[F_A, X] + (t^2 - t)\{A, [\mathrm{d} + A, X]\}\\ &= [tF_A, X] + [(t^2 - t)A^2, X] + (t-1)\{tA, [\mathrm{d}, X]\},\end{aligned}$$

where we used $\{A, [A, X]\} = [A^2, X]$. Adding $(t-1)\{\mathrm{d}, [\mathrm{d}, X]\} = 0$ yields (33).)

With Grassmann numbers θ_j as above we introduce

$$\mathcal{A} \equiv \theta_0 A + \sum_{j=1}^{k}(t-1)\theta_j[\mathrm{d}, X_j], \quad \mathcal{F}_\pm \equiv F_{tA} \pm \mathcal{A} \tag{34}$$

and the operator

$$\Delta = \theta_0 \partial_t + \sum_{j=1}^{k} \theta_j \hat{\mathcal{L}}_{X_j}, \tag{35}$$

and we calculate $\Delta(\mathcal{F}_+)$ which we write as $(\cdot)_1 + (\cdot)_2$. Using $\partial_t(F_{tA}) = \{\mathrm{d} + tA, A\}$,

$$(\cdot)_1 \equiv \theta_0 \partial_t(\mathcal{F}_+) = \{\mathrm{d} + tA, \theta_0 A\} + \sum_{j=1}^{k} \theta_0 \theta_j [\mathrm{d}, X_j].$$

The relations above and the definition (20) of $\hat{\mathcal{L}}_X$ imply

$$\hat{\mathcal{L}}_X(A) = [\mathrm{d}, X], \qquad \hat{\mathcal{L}}_X(F_{tA}) = (t-1)\{\mathrm{d} + tA, [\mathrm{d}, X]\}$$

and $\hat{\mathcal{L}}_X(Y) = -[Y, X] = [X, Y]$ if Y is independent of A, thus

$$\begin{aligned}
(\cdot)_2 &\equiv \sum_{i=1}^{k} \theta_i \hat{\mathcal{L}}_{X_i}(\mathcal{F}_+) \\
&= \sum_{i=1}^{k} \theta_i \Big((t-1)\{\mathrm{d} + tA, [\mathrm{d}, X_i]\} + \theta_0 [\mathrm{d}, X_i] \\
&\quad + \sum_{j=1}^{k} (t-1)\theta_j [X_i, [\mathrm{d}, X_j]]\Big) \\
&= \left\{\mathrm{d} + tA, \sum_{i=1}^{k} (t-1)\theta_i [\mathrm{d}, X_i]\right\} + \sum_{i=1}^{k} \theta_i \theta_0 [\mathrm{d}, X_i] + \Xi,
\end{aligned}$$

where we introduced $\Xi = \sum_{i,j=1}^{k} \theta_i \theta_j (t-1)[X_i, [\mathrm{d}, X_j]]$, or equivalently (using $\{\theta_i, \theta_j\} = 0$, and the Jacobi identity and antisymmetry of $[\cdot, \cdot]$)

$$\Xi = \sum_{\substack{i,j=1 \\ i<j}}^{k} (t-1)\theta_i \theta_j [\mathrm{d}, [X_i, X_j]]. \tag{36}$$

Putting this together we see that the second terms in $(\cdot)_1$ and $(\cdot)_2$ cancel each other, and we obtain

$$\Delta(\mathcal{F}_+) = \{\mathrm{d} + tA, \mathcal{A}\} + \Xi. \tag{37}$$

To proceed it is convenient to use another Grassmann number $\theta \equiv \theta_{k+1}$ which satisfies $\theta \mathcal{A} = -\mathcal{A}\theta$. This allows to rewrite (37) as $\theta \Delta(\mathcal{F}_+) = [\theta(\mathrm{d} + tA), \mathcal{A}] + \theta \Xi$, or equivalently (since $[\theta(\mathrm{d} + tA), F_{tA}] = 0$)

$$\theta \Delta(\mathcal{F}_+) = [\theta(\mathrm{d} + tA), \mathcal{F}_+] + \theta \Xi.$$

Since $\theta \Delta \mathcal{F}_+ = \theta \Delta(\mathcal{F}_+) + \mathcal{F}_+ \theta \Delta$, this implies

$$
\begin{aligned}
\theta\Delta((\mathcal{F}_+)^n) &= \sum_{v=0}^{n-1}(\mathcal{F}_+)^{n-1-v}\theta\Delta(\mathcal{F}_+)(\mathcal{F}_+)^v \\
&= \sum_{v=0}^{n-1}(\mathcal{F}_+)^{n-1-v}([\theta(\mathrm{d}+tA),\mathcal{F}_+]+\theta\Xi)(\mathcal{F}_+)^v \\
&= [\theta(\mathrm{d}+tA),(\mathcal{F}_+)^n]+\sum_{v=0}^{n-1}(\mathcal{F}_+)^{n-1-v}\theta\Xi(\mathcal{F}_+)^n
\end{aligned}
$$

as $\sum_{v=0}^{n-1}(\mathcal{F}_+)^{n-1-v}[\cdot,\mathcal{F}_+](\mathcal{F}_+)^v = [\cdot,(\mathcal{F}_+)^n]$. To get rid of θ again we use $\mathcal{F}_+\theta = \theta\mathcal{F}_-$ (cf. (34)), thus

$$
\Delta((\mathcal{F}_+)^n) = (\mathrm{d}+tA)(\mathcal{F}_+)^n - (\mathcal{F}_-)^n(\mathrm{d}+tA) + \sum_{v=0}^{n-1}(\mathcal{F}_-)^{n-1-v}\Xi(\mathcal{F}_+)^v. \tag{38}
$$

We now apply integration $\int \mathrm{d}\theta_k\cdots\mathrm{d}\theta_0$ to this equation. In the following calculation we keep in mind that we can always forget about terms containing higher powers than 1 of at least one of the θ_i's. Thus,

$$
\begin{aligned}
&\int \mathrm{d}\theta_k\cdots\mathrm{d}\theta_0\Delta((\mathcal{F}_+)^n) \\
&= \int \mathrm{d}\theta_k\cdots\mathrm{d}\theta_0\,\theta_0\partial_t\left(F_{tA}+\sum_{j=1}^{k}(t-1)\theta_j[\mathrm{d},X_j]\right)^n \\
&\quad + \sum_{i=1}^{k}\hat{\mathcal{L}}_{X_i}\int \mathrm{d}\theta_k\cdots\mathrm{d}\theta_0\,\theta_i\left(F_{tA}+\theta_0 A+\sum_{\substack{j=1\\ j\neq i}}^{k}(t-1)\theta_j[\mathrm{d},X_j]\right)^n .
\end{aligned}
$$

Using

$$
\int \mathrm{d}\theta_k\cdots\mathrm{d}\theta_0\theta_j = (-)^j\int \mathrm{d}\theta_k\cdots\mathrm{d}\cancel{\theta}_j\cdots\mathrm{d}\theta_0 \quad j=0,1,\ldots,k,
$$

the definition (27a) of v_{2n-k}^{k-1} and introducing the k-chains

$$
\beta_{2n-k}^{k}(X_1,\ldots,X_k;A,t) = \int \mathrm{d}\theta_k\cdots\mathrm{d}\theta_1\left(F_{tA}+\sum_{j=1}^{k}(t-1)\theta_j[\mathrm{d},X_j]\right)^n , \tag{39}
$$

we can write this

$$
\begin{aligned}
&\int \mathrm{d}\theta_k\cdots\mathrm{d}\theta_0\Delta((\mathcal{F}_+)^n) \\
&= \partial_t\beta_{2n-k}^{k}(X_1,\ldots,X_k;A,t) \\
&\quad + \sum_{j=1}^{k}(-)^j\hat{\mathcal{L}}_{X_j}v_{2n-k}^{k-1}(X_1,\ldots,\cancel{X}_j,\ldots,X_k;A,t).
\end{aligned}
$$

Moreover, since $\int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 (\mathcal{F}_\pm)^n = (\pm)^{k+1} \nu^k_{2n-k-1}(X_1, \ldots, X_k; A, t)$ (cf. (27a) and (34)), we have

$$\int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0((\mathrm{d} + tA)(\mathcal{F}_+)^n - (\mathcal{F}_-)^n(\mathrm{d} + tA))$$
$$= [\mathrm{d} + tA, \nu^k_{2n-k-1}(X_1, \ldots, X_k; A, t)]_\sigma, \quad \sigma = (-)^k.$$

Note that this formula remains also true for $k = n$ if we set v^n_{n-1} to zero. Finally (we use $\mathcal{F}_-\theta_i = \theta_i\mathcal{F}_+$; cf. (36)),

$$\int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \sum_{v=0}^{n-1} (\mathcal{F}_-)^{n-1-v} \Xi (\mathcal{F}_+)^v$$
$$= \sum_{\substack{i,j \\ i<j}} \sum_{v=0}^{n-1} \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \theta_i (\mathcal{F}_+)^{n-1-v}(t-1)\theta_j[\mathrm{d}, [X_i, X_j]](\mathcal{F}_+)^v$$
$$= \sum_{\substack{i,j \\ i<j}} \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \theta_i \left(F_{tA} + \theta_0 A + (t-1)\theta_j[\mathrm{d}, [X_i, X_j]] + \sum_{\substack{l=1 \\ l\neq i,j}}^{k} (t-1)\theta_l[\mathrm{d}, X_l] \right)^n$$
$$= \sum_{\substack{i,j \\ i<j}} (-)^{i+j} \nu^{k-1}_{2n-k}([X_i, X_j], X_1, \ldots, \cancel{X_i}, \ldots, \cancel{X_j}, \ldots, X_k; A, t),$$

where we used

$$\int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \theta_i = (-)^{i+j} \int \mathrm{d}\theta_k \cdots \mathrm{d}\cancel{\theta_i} \cdots \mathrm{d}\cancel{\theta_j} \cdots \mathrm{d}\theta_1 \, \mathrm{d}\theta_j \, \mathrm{d}\theta_0.$$

Using the definitions (23) and (24) we can therefore write $\int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0$ (38) as

$$\hat{\delta}\nu^{k-1}_{2n-k} + [\mathrm{d} + tA, \nu^k_{2n-k-1}]_\sigma = \partial_t \beta^k_{2n-k}, \quad \sigma = (-)^k, \tag{40}$$

where we suppressed the common argument $(X_1, \ldots, X_k; A, t)$. Our descent equations for $k = 1, \ldots, n-1$ are obtained by applying integration $\int_0^1 \mathrm{d}t$ to (40). We first note that for $k = 1, \ldots, n-1$, $\beta^k_{2n-k}(X_1, \ldots, X_k; A, t)$ is zero for $t = 1$ (trivially) and $t = 0$ (since $F_0 = 0$), thus $\int_0^1 \mathrm{d}t \, \partial_t \beta^k_{2n-k} = 0$. Thus with the definitions (28a) and $\hat{\delta} = \delta - \mathcal{J}$,

$$\delta\omega^{k-1}_{2n-k} - \mathcal{J}\omega^{k-1}_{2n-k} + [\mathrm{d}, \omega^k_{2n-k-1}]_\sigma + [A, \tilde{\omega}^k_{2n-k-1}]_\sigma = 0,$$
$$\sigma = (-)^k, \quad k = 1, 2, \ldots, n-1 \tag{41}$$

equivalent to (29b). For $k = n$, Eq. (40) remains true with v^n_{n-1} set to zero, thus

$$\delta\omega^{n-1}_n - \mathcal{J}\omega^{n-1}_n = N^n_n, \tag{42}$$

where $N_n^n \equiv \int_0^1 \mathrm{d}t\, \partial_t \beta_n^n$ is nonzero,

$$N_n^n(X_1, \dots, X_n) = (-)^{n+1} \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \left(\sum_{i=1}^n \theta_i\, \mathrm{d}(X_i) \right)^n . \tag{43}$$

To complete the proof of (29c), we will show below that N_n^n equals $-\mathrm{d}(\omega_{n-1}^n)$.

(c) Here $k = n, n+1, \dots, 2n-1$.

We introduce

$$\mathcal{V} = \sum_{i=1}^k \theta_i X_i, \qquad \tilde{\mathcal{F}}_\pm = \pm t\mathrm{d}(\mathcal{V}) + (t^2 - t)\mathcal{V}^2 + \theta_0 \mathcal{V} \tag{44}$$

and the operator

$$\tilde{\Delta} = \theta_0 \partial_t + t\hat{I}, \qquad \hat{I} = \sum_{i=1}^k \theta_i [X_i, \cdot]. \tag{45}$$

We observe

$$\mathcal{V}^2 = \tfrac{1}{2} \sum_{i,j=1}^k \theta_i\theta_j [X_i, X_j] = \sum_{\substack{i,j=1 \\ i<j}}^k \theta_i\theta_j [X_i, X_j]$$

implying $\hat{I}(\mathcal{V}) = 2\mathcal{V}^2$. Similarly,

$$\hat{I}(\mathrm{d}(\mathcal{V})) = \tfrac{1}{2} \sum_{i,j=1}^k \theta_i\theta_j([X_i,\, \mathrm{d}(X_j)] - [X_i,\, \mathrm{d}(X_j)]) = \mathrm{d}(\mathcal{V}^2),$$

and

$$\begin{aligned} \hat{I}(\mathcal{V}^2) &= \sum_{l=1}^k \sum_{\substack{i,j=1 \\ i<j}}^k \theta_l\theta_i\theta_j [X_l, [X_i, X_j]] \\ &= \tfrac{1}{6} \sum_{i,j,l=1}^k \theta_l\theta_i\theta_j ([X_l, [X_i, X_j]] + [X_i, [X_j, X_l]] + [X_j, [X_l, X_i]] \\ &= 0, \end{aligned}$$

where we used $\theta_l\theta_i\theta_j = \theta_i\theta_j\theta_l$, etc. and the Jacobi identity for $[\cdot, \cdot]$. Thus

$$t\hat{I}(\tilde{\mathcal{F}}_+) = t^2\, \mathrm{d}(\mathcal{V}^2) - 2t\theta_0 \mathcal{V}^2$$

(note that $\hat{I}\theta_0 = -\theta_0\hat{I}$). Obviously $\theta_0\partial_t(\tilde{\mathcal{F}}_+) = \theta_0\, \mathrm{d}(\mathcal{V}) + (2t-1)\theta_0\mathcal{V}^2$. Combining the last two equations we obtain

$$\tilde{\Delta}(\tilde{\mathcal{F}}_+) = [\mathrm{d}, \tilde{\mathcal{F}}_+] + \tilde{\Xi}, \quad \tilde{\Xi} = t\mathrm{d}(\mathcal{V}^2) - (t^2 - t)\hat{I}(\mathcal{V}^2) - \theta_0\mathcal{V}^2, \tag{46}$$

where we added terms $\mathrm{d}(t\mathrm{d}(\mathcal{V})) = -2(t^2-t)\hat{I}(\mathcal{V}^2) = 0$ and used $\mathrm{d}(\tilde{\mathcal{F}}_+) = [\mathrm{d}, \tilde{\mathcal{F}}_+]$. Similarly as in (b) above this implies

$$\tilde{\Delta}((\tilde{\mathcal{F}}_+)^n) = \mathrm{d}(\tilde{\mathcal{F}}_+)^n - (\tilde{\mathcal{F}}_-)^n\,\mathrm{d} + \sum_{v=0}^{n-1}(\tilde{\mathcal{F}}_-)^{n-1-v}\tilde{\Xi}(\tilde{\mathcal{F}}_+). \tag{47}$$

Noting that our definition (27b) can be written as $v^k_{2n-k-1} = (-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0((\tilde{\mathcal{F}}_+)^n$, we now apply $(-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0$ to this equation. The following calculation is similar to the one above in (b),

$$\begin{aligned}
&(-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \tilde{\Delta}((\mathcal{F}^n_+)) \\
&= \partial_t \tilde{\beta}^k_{2n-k}(X_1, \ldots, X_k; t) \\
&+ \sum_{i=1}^{k}(-)^i t[X_i, v^{k-1}_{2n-k}(X_1, \ldots, \not{X}_i, \ldots, X_k; t)],
\end{aligned}$$

where we introduced

$$\tilde{\beta}^k_{2n-k}(X_1, \ldots, X_k; t) = (-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_1 (t\mathrm{d}(\mathcal{V}) + (t^2-t)\mathcal{V}^2)^n \tag{48}$$

and used the definition (27b). Note that this remains true also for $k = n$ if we set v^{n-1}_n to zero. Similarly as in (b),

$$\begin{aligned}
&(-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0(\mathrm{d}(\tilde{\mathcal{F}}_+)^n - (\tilde{\mathcal{F}}_-)^n\,\mathrm{d}) \\
&= [\mathrm{d}, v^k_{2n-k-1}(X_1, \ldots, X_k; t)]_\sigma, \quad \sigma = (-)^k.
\end{aligned}$$

To calculate the last term we write

$$\hat{\Xi} = \sum_{\substack{i,j=1\\ i<j}}^{k} \theta_i \left(t\theta_j\,\mathrm{d}([X_i, X_j]) + \sum_{\substack{l=1\\ l\neq i,j}}^{k}(t^2-t)\theta_j\theta_l[[X_i, X_j], X_l] + \theta_0\theta_j[X_i, X_j] \right),$$

and similarly as in (b),

$$\begin{aligned}
&(-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0 \sum_{v=0}^{n-1}(\tilde{\mathcal{F}}_-)^{n-1-v}\tilde{\Xi}(\tilde{\mathcal{F}}_+)^v \\
&= \sum_{\substack{i,j\\ i<j}}(-)^{i+j} v^{k-1}_{2n-k}([X_i, X_j], X_1, \ldots, \not{X}_i, \ldots, \not{X}_j, \ldots, X_k; t).
\end{aligned}$$

Recalling the definitions (23) and (24) we can therefore write $(-)^n \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0$ (47) as

$$\delta v^{k-1}_{2n-k} + [\mathrm{d}, v^k_{2n-k-1}]_\sigma = \partial_t \tilde{\beta}^k_{2n-k} + \mathcal{J} t v^{k-1}_{2n-k}, \quad \sigma = (-)^k, \tag{49}$$

where we suppressed the common argument $(X_1, \ldots, X_k; t)$ (we used that $\mathcal{L}_{X_j}$ acting on anything independent of A gives zero). Again the descent equations are obtained by integrating $\int_0^1 \mathrm{d}t$ this equation. Since for $k > n$, $\tilde{\beta}^k_{2n-k}$ is zero for $t = 0$ and $t = 1$ we get

$$\delta\omega_{2n-k}^{k-1} + \mathrm{d}(\omega_{2n-k-1}^{k}) = \mathcal{J}\tilde{\omega}_{2n-k-1}^{k-1} \tag{50}$$

proving (29d) for $k = n+1, \ldots, 2n-1$. Obviously this equation remains also true for $k = 2n$ if we set $\mathrm{d}(\omega_{-1}^{2n}) = 0$. This implies (29e).

For $k = n$, (49) remains true if we set v_n^{n-1} to zero. Moreover, it is easy to see that $\int_0^1 \mathrm{d}t\,\partial_t\tilde{\beta}_n^n = -N_n^n$ as defined in (43) (cf. (48)), thus

$$\mathrm{d}(\omega_{n-1}^{n}) = -N_n^n. \tag{51}$$

Combining this with Eq. (42) gives (29c) which completes the proof of the theorem.

The corollary then follows trivially from our definitions, especially the defining relations (13) of integrations on GDAPI.

4. Final remarks

Given a GDA $(\Omega,\ \mathrm{d})$ one can consider an $\mathbb{N}_0$-graded vector space $C = \oplus_{k=0}^{\infty} C^{(k)}$ of polynomial maps

$$c^k : \underbrace{\Omega^{(0)} \times \cdots \times \Omega^{(0)}}_{(k-1)\ \text{times}} \times \Omega^{(1)} \to \mathbb{C},$$

$$(X_1, \ldots, X_k, A) \mapsto c^k(X_1, \ldots, X_k; A).$$

which are multilinear and antisymmetric in the first k arguments, with the operator $\delta : C^{(k)} \to C^{(k+1)}$ defined as in (23). Since $\delta^2 = 0$, (C, δ) is a graded differential complex. A map $c^k \in C^{(k)}$ is called a k-cocycle if $\delta c^k = 0$, and a k-cocycle c^k is nontrivial if it is not a k-boundary, i.e. not of the form δb^{k-1} with $b^{k-1} \in C^{(k-1)}$. Anomalies in quantum gauge theories are nontrivial cocycles, e.g. the axial anomalies and Schwinger terms are nontrivial 1- and 2-cocycles, respectively [J].

The result of this paper in Section 3 provides an explicit construction of k-cocycles

$$c_{2n-k-1}^{k} = \int \omega_{2n-k-1}^{k} \tag{52}$$

for arbitrary GDA $(\Omega,\ \mathrm{d})$ with integration $\int$. The cocycle property is a trivial consequence of our generalized descent equations and basis properties of integration. Whether or not c_{2n-k-1}^{k} is nontrivial depends on the integration $\int$ and the precise definition of $C^{(k-1)}$ – in most cases will be trivial or even zero. Well-known examples for (52) being nontrivial is for the de Rham GDA $(\Omega_{\mathrm{dR},2n-k-1}, \mathrm{d})$ with integration $\int_{\mathbb{R}^{2n-k-1}} \mathrm{tr}_N$, or more generally, the de Rham GDA $(\Omega_{\mathrm{dR},d}, \mathrm{d})$, d large enough, with integration $\int_M \mathrm{tr}_N$ over a $(2n-k-1)$-dimensional submanifold M of $\mathbb{R}^d$. According to our discussion in Section 2, these examples immediately generalize to universal Rham GDA (i.e. Example 2; the hats in the following are to indicate that we are considering universal de Rham forms). We conjecture that all the universal cocycles $\hat{c}_{2n-k-1}^{k}$ for $(\hat{\Omega}_{\mathrm{dR},2n-k-1}, \hat{\mathrm{d}}, \hat{\int})$ are nontrivial since this is known to be true for $k = 2$ and $n = 2$ and $n = 3$. Indeed, $\hat{c}_1^2$ is proportional to the Lundberg cocycle [Lb] known to be nontrivial, and $\hat{c}_3^2$ is proportional to the Mickelsson–Rajeev cocycle which is also nontrivial [MR].

To have one explicit example and also demonstrate how formulas can be written without using Grassmann numbers, we finally give a more explicit expression for the universal Schwinger terms generalizing the Mickelsson–Rajeev cocycle to higher dimensions,

$$\begin{aligned}&\hat{c}_{2n-3}^2(\hat{X}_1, \hat{X}_2; \hat{A})\\ &= \frac{3}{c_{2n-3}} \sum_{\substack{i_\nu \geq 0 \\ \sum_\nu i_\nu = n-3}} (i)^2 \int_0^1 \mathrm{d}t (t-1)^2 \\ &\quad \times \mathrm{Tr}_C((\hat{F}_{tA})^{i_1} \hat{A} (\hat{F}_{tA})^{i_2} [\varepsilon, \hat{X}_1] (\hat{F}_{tA})^{i_3} [\varepsilon, \hat{X}_2] (\hat{F}_{tA})^{i_4} - (\hat{X}_1 \longleftrightarrow \hat{X}_2)),\\ &\quad n \geq 3, \end{aligned} \tag{53}$$

where $\hat{F}_{tA} = \mathrm{i}\{\varepsilon, \hat{A}\}t + \hat{A}^2 t^2$ (we used cyclicity under conditional Hilbert space trace; for notation see Section 2); the $\hat{X}_i$ and $\hat{A}$ here are bounded operators on a Hilbert space $\mathcal{H}$ such that $[\varepsilon, \hat{X}_i]$ and $[\varepsilon, \hat{A}]$ both are in the Schatten class $\mathcal{B}_{2n-2}$ and $\{\varepsilon, \hat{A}\} \in \mathcal{B}_{n-1}$. Restricting to operators $\hat{A} = (F - \varepsilon)$ where $F = F^{-1} = F^*$ is a grading operator (as usually done in this context [L1]), we have $\hat{F}_{tA} = (t^2 - t)(F - \varepsilon)^2$, and (53) should be proportional to the universal Schwinger terms given in [FT].

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